Module-3: Schwarz Lemma and Its Applications

The following theorem is an application of Rouche's theorem.

Theorem 1. (Hurwitz Theorem)
Let $\{f_n(z)\}$ be a sequence of analytic functions defined on a domain D such that $f_n(z) \neq 0 \; \forall z \in D,\; n = 1, 2, \ldots$. Suppose that $\{f_n(z)\}$ converges uniformly to a function $f(z)$ on every compact subset of D. Then either $f(z) \equiv 0$ or $f(z) \neq 0$ for all $z \in D$.

Proof. Suppose that $f(z) \not\equiv 0$ in D. We have to show that $f(z) \neq 0$ for all $z \in D$. If possible, we assume that there is some $z_0 \in D$ for which $f(z_0) = 0$. As zeros of an analytic function are isolated, there is a deleted neighbourhood of z_0, namely, $N_\delta(z_0)$, in which the function $f(z)$ is not zero. That is,

$$f(z) \neq 0, \quad for \; z \in \; 0 <| z - z_0 | < \delta, \; \delta > 0.$$

Let $\delta' > 0$ be such that $\delta' < \delta$. Then $f(z) \neq 0$ on the punctured disc $0 <| z - z_0 | < \delta'$. Let C be the circle having centre at z_0 and radius δ' and $\varepsilon = \min\{| f(z) | : z \in C\}$. C being a compact subset of D, the sequence $f_n(z)$ converges uniformly to $f(z)$ on C. Therefore, given $\varepsilon > 0$, there is a positive integer N such that for $z \in C$

$$| f_n(z) - f(z) | < \varepsilon \; \forall \; n > N.$$

Noting that $\varepsilon \leq | f(z) |$ whenever $z \in C$, from above we obtain

$$| f_n(z) - f(z) | < \varepsilon \leq | f(z) | \;\; \forall \; n > N',$$

and for all z on C. Hence, by Rouche's theorem we can say that the functions $f(z)$ and $[f_n(z) - f(z)] + f(z) = f_n(z)$ have the same number of zeros inside C. By assumption,

$f(z)$ has a zero at z_0. Therefore, $f_n(z)$ must also have a zero inside C, a contradiction to the hypothesis. Thus, we can conclude that $f(z) \neq 0$ for all $z \in D$ unless the function is identically zero. This proves the theorem. □

Alternative statement of Hurwitz Theorem

Let $\{f_n(z)\}$ be a sequence of functions analytic inside and on a simple closed contour C, and suppose that $\{f_n(z)\}$ converges uniformly to a function $f(z)$ inside and on C. If $f(z)$ has no zeros on C, then the number of zeros of $f(z)$ inside C is equal to the number of zeros of $f_n(z)$ inside C for sufficiently large n.

We now come to an important geometric property of analytic functions that arises when we consider them as mappings (that is, mapping regions in the complex plane to the complex plane).

A mapping in the complex plane is said to be an open mapping if it maps open sets into open sets.

Theorem 2. (Open Mapping Theorem)

A nonconstant analytic function maps open sets onto open sets.

Proof. Suppose that the function $w = f(z)$ is analytic at $z = z_0$. We have to show that the image of every sufficiently small neighbourhood of z_0 in the z-plane contains a neighbourhood of $w_0 = f(z_0)$ in the w-plane. We choose a positive number δ such that $f(z) - w_0$ is analytic in the disk $|z - z_0| \leq \delta$ and contains no zero on the circle $|z - z_0| = \delta$. Let m be the minimum of $|f(z) - w_0|$ on the circle $|z - z_0| = \delta$ (see Fig.1). We shall show that the image of the disk $|z - z_0| < \delta$ under $f(z)$ contains the disk $|w - w_0| < m$. Let w_1 be an arbitrary but fixed point in the disk $|w - w_0| < m$. Then on the circle $|z - z_0| = \delta$ we have

$$|w_0 - w_1| < m \leq |f(z) - w_0|.$$

Therefore, by Rouche's theorem, the functions $f(z) - w_0$ and $(f(z) - w_0) + (w_0 - w_1) = f(z) - w_1$ has the same number of zeros in $|z - z_0| < \delta$. Since the function $f(z) - w_0$ has at least one zero at z_0, the function $f(z) - w_1$ has at least one zero. This means that $f(z) = w_1$ at least once. Since w_1 is arbitrary, the image of the disk $|z - z_0| < \delta$ must contain all points in the disk $|w - w_0| < m$. This proves the theorem. □

Corollary 1. *A nonconstant analytic function maps a domain onto a domain.*

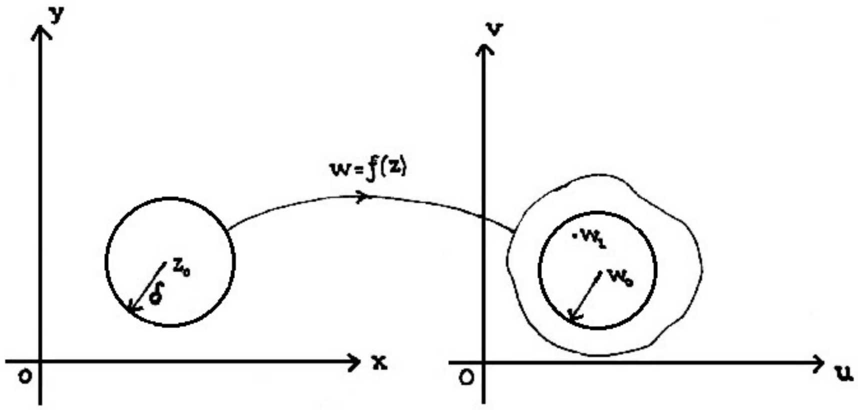

Fig. 1:

Proof. We recall that a domain is an open connected set. Let $f(z)$ be a nonconstant analytic function. We know that a continuous image of a connected set is connected. Also in view of Theorem 2, it follows that $f(z)$ maps an open set onto an open set. Thus, $f(z)$, being an analytic function, is continuous and hence the result follows. □

Corollary 2. *If $f(z)$ is analytic and univalent in a domain D, then $f'(z) \neq 0$ in D.*

Proof. If possible, we assume that $f'(z_0) = 0$ for some z_0 in D. Since $f(z)$ is analytic in D, it has a Taylor series expansion about z_0 of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Now $f'(z_0) = 0$ implies that $a_1 = 0$ and hence,

$$f(z) - f(z_0) = \sum_{n=2}^{\infty} a_n (z - z_0)^n$$

has a zero of order k at z_0 where $k \geq 2$ is an integer. By hypothesis $f(z)$ is univalent and so $f(z)$ is a nonconstant function. Obviously, $f(z) - f(z_0)$ is analytic in D. Since the zeros of an analytic function are isolated, we can find a neighbourhood $| z - z_0 | \leq \delta, \ \delta > 0$, in which $f(z) - f(z_0)$ has no zero except z_0. Thus $f(z) - f(z_0) \neq 0$ on the circle $| z - z_0 | = \delta$, and hence

$$m = \inf\{| f(z) - f(z_0) | : \ z \in | z - z_0 | = \delta\}$$

is positive. Then, for any complex number c, $0 < |c| < m$, we have

$$|c| < |f(z) - f(z_0)|$$

$$\text{or} \quad |[f(z) - f(z_0) - c] - [f(z) - f(z_0)]| < |f(z) - f(z_0)|.$$

So, by Rouche's theorem $f(z) - f(z_0) - c$ and $f(z) - f(z_0)$ have the same number of zeros inside $|z - z_0| = \delta$. Since $f(z) - f(z_0)$ has a zero of order k ($k \geq 2$) inside this circle, it therefore, follows that $f(z) - f(z_0) - c$ has two or more zeros inside this circle. This means that the equation $f(z) = f(z_0) + c$ is satisfied at two or more points. This contradicts the fact that f is univalent in D. Thus, $f'(z) \neq 0$ for all points in D. This proves the corollary. □

Corollary 3. *If $f(z)$ is analytic and univalent in a domain D, then the inverse map $z = f^{-1}(w)$ is analytic in $f(D)$.*

Proof. Since $f(z)$ is analytic and univalent in D, by Corollary 2, $f'(z) \neq 0$ in D. Let w_0 be a fixed point in $f(D)$. Then there exist a unique $z_0 \in D$ such that $f(z_0) = w_0$. Let f^{-1} be the inverse of f. Then for any $w \in f(D)$,

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}$$

is well defined for all w in a deleted neighbourhood of w_0. Proceeding to the limit as $w \to w_0$ we obtain from above

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}.$$

$w_0 \in f(D)$ being arbitrary, it follows from above that f^{-1} is analytic in $f(D)$. This proves the corollary. □

Example 1. *Let f be analytic in a domain D. If any one of Re f, Im f, $|f|$, or Arg f is constant, then show that f is constant.*

Solution. *Since any one of Re f, Im f, $|f|$, or Arg f is constant, $f(D)$ would be a subset of either the real axis, or the imaginary axis, or a circle or a line with constant argument, respectively. Since none of them forms an open set, by open mapping theorem, it follows that f is constant.*

If $f(z)$ is nonconstant analytic function in $|z| \leq R$ and $|f(z)| \leq M$ on $|z| = R$, then the maximum modulus principle says that $|f(z)| < M$ for $|z| < R$. We now develop methods to improve this bound inside the disc.

Theorem 3. (Schwarz Lemma)
Let $f(z)$ be analytic in $|z| < 1$, with a zero of order n at the origin. Suppose that $|f(z)| \leq 1$ for all z in $|z| < 1$. Then,

$$|f(z)| \leq |z|^n, \quad |z| < 1, \tag{1}$$

$$|f^{(n)}(0)| \leq n!. \tag{2}$$

Further, equality holds in (1) for some $z \neq 0$ or in (2) if and only if $f(z)$ is of the form

$$f(z) = cz^n, \quad |c| = 1. \tag{3}$$

Proof. Since $f(z)$ is analytic in $|z| < 1$, by its Taylor expansion we obtain

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 1,$$

where $a_k = \frac{f^{(k)}(0)}{k!}$. Also, $f(z)$ has a zero of order n at the origin implies that $f(0) = f'(0) = f''(0) = \ldots f^{(n-1)}(0) = 0$. So from above we get

$$\begin{aligned} f(z) &= \sum_{k=n}^{\infty} a_k z^k \\ &= \frac{f^{(n)}(0)}{n!} z^n + \frac{f^{(n+1)}(0)}{(n+1)!} z^{n+1} + \ldots, \quad |z| < 1. \end{aligned}$$

For $z = 0$, (1) is the hypothesis. Let $z \neq 0$. Then from above we obtain

$$\frac{f(z)}{z^n} = \frac{f^{(n)}(0)}{n!} + \frac{f^{(n+1)}(0)}{(n+1)!} z + \ldots, \quad 0 < |z| < 1.$$

The series on the right side converges for $z = 0$. Let us define

$$g(z) = \begin{cases} \frac{f(z)}{z^n}, & z \neq 0, \\ \frac{f^{(n)}(0)}{n!}, & z = 0. \end{cases}$$

Now $g(z)$ is analytic in $|z| < 1$. Let C denote the circle $|z| = r$, where $0 < r < 1$. Then by the maximum modulus principle we have

$$|g(z)| \leq \max_{|z|=r} |g(z)| = \max_{|z|=r} \left|\frac{f(z)}{z^n}\right| \leq \frac{1}{r^n}, \quad |z| \leq r.$$

The above inequality is true for all $r < 1$, letting $r \to 1$, we obtain

$$\frac{| f(z) |}{| z |^n} \leq 1 \ \ i.e. \ \ | f(z) | \leq | z |^n .$$

Since $g(0) = f^{(n)}(0)/n!$ and $| g(0) | \leq 1$, we obtain (2).

In case $| f(z_0) | = | z_0 |^n$ for some z_0 in $0 < | z_0 | < R$, then $g(z_0) = 1$. It means that $| g |$ attains its maximum at the interior point z_0. This is only possible if $g(z)$ is a constant, say, $g(z) = c$ or $f(z) = cz^n$ for some constant c with $| c | = 1$, which is (3).

The case $f^{(n)}(0) = n!$ can be proved similarly. This proves the theorem. □

The following corollaries are the generalization of the Schwarz theorem.

Corollary 4. *Let $f(z)$ be analytic in $| z | < R$, with a zero of order n at the origin. Suppose that $| f(z) | \leq M$ for all z in $| z | < R$. Then,*

$$| f(z) | \leq \frac{M | z |^n}{R^n}, \ \ | z | < R, \tag{4}$$

$$| f^{(n)}(0) | \leq \frac{Mn!}{R}. \tag{5}$$

Further, equality holds in (4) for some $z \neq 0$ or in (5) if and only if $f(z)$ is of the form

$$f(z) = \frac{Mcz^n}{R^n}, \ \ | c | = 1.$$

Corollary 5. *Let $f(z)$ be analytic in $| z - z_0 | < R$, with a zero of order n at z_0. Suppose that $| f(z) | \leq M$ for all z in $| z - z_0 | < R$. Then,*

$$| f(z) | \leq \frac{M | z - z_0 |^n}{R^n}, \ \ | z - z_0 | < R, \tag{6}$$

$$| f^{(n)}(z_0) | \leq \frac{Mn!}{R}. \tag{7}$$

Further, equality holds in (6) for some $z \neq z_0$ or in (7) if and only if $f(z)$ is of the form

$$f(z) = \frac{Mc | z - z_0 |^n}{R^n}, \ \ | c | = 1.$$

Some results

1. Let $f(z)$ be analytic in $| z | < 1$ and $| f(z) | \leq 1$ on $| z | < 1$. Then

$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0} z} \right|,$$

for any $z, \ z_0$ inside the unit disc.

2. Let $f(z)$ be analytic in $| z | < 1$ and $| f(z) | \leq 1$ on $| z | < 1$. Then

(i) $| f(z) | \leq \frac{|f(0)|+|z|}{1+|f(0)||z|}$;

(ii) $| f'(z) | \leq \frac{1-|f(z)|^2}{1-|z|^2}$. This result is known as Schwarz-Pick lemma.

Example 2. *Suppose that* $f(z)$ *is analytic for* $|z| \leq 1$ *such that* $|f(z)| < 1$ *for* $|z| = 1$*. Show that* $f(z) = z^n$ *has exactly* n *solutions in* $|z| < 1$.

Solution. *Let* $g(z) = -z^n$*. For* $|z| = 1$*, we have* $|f(z)| < 1 = |-z^n| = |g(z)|$ *. Hence by Rouche's theorem, we can say that* $g(z)$ *and* $f(z) + g(z) = f(z) - z^n$ *has same number of zeros in* $|z| < 1$*. Since* $g(z)$ *has* n *zeros at origin, it follows that* $f(z) = z^n$ *has exactly* n *solutions in* $|z| < 1$.

Example 3. *Let* f *be analytic in* $|z| < 1$ *such that* $|f(z)| \leq 1$ *and* $f(1/3) = 0$*. Then, estimate* $|f(1/7)|$ *.*

Solution. *Since* f *is analytic in* $|z| < 1$ *and* $|f(z)| \leq 1$ *for* $|z| < 1$*, we have*

$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0} z} \right|,$$

for any z*,* z_0 *inside the unit disc. Taking* $z = 1/7$ *and* $z_0 = 1/3$ *we obtain*

$$\left| \frac{f(1/7) - f(1/3)}{1 - \overline{f(1/3)} f(1/7)} \right| \leq \left| \frac{1/7 - 1/3}{1 - 1/21} \right|$$

$$i.e. \quad |f(1/7)| \leq 1/5.$$